Structure functions (Kolmogorv, 1941b)

2nd-order structure function:

$$D_{ij}\left(\boldsymbol{r},\boldsymbol{x},t\right)=\overline{\left[\tilde{u}_i\left(\boldsymbol{x}+\boldsymbol{r},t\right)-\tilde{u}_i\left(\boldsymbol{x},t\right)\right]\left[\tilde{u}_j\left(\boldsymbol{x}+\boldsymbol{r},t\right)-\tilde{u}_j\left(\boldsymbol{x},t\right)\right]},$$

which is a covariance of the velocity difference between $\boldsymbol{x}+\boldsymbol{r}$ and $\boldsymbol{x}$.

Local isotropy: For $|\boldsymbol{r}|\ll\mathcal{L}$, D_{ij} is a function of $\boldsymbol{r}$ only independent of $\boldsymbol{x}$. A second-order isotropic tensor can be written $A\delta_{ij}+Br_ir_j$, thus,

$$D_{ij}\left(\boldsymbol{r},t\right)=D_{NN}\left(r,t\right)\delta_{ij}+\left[D_{LL}\left(r,t\right)-D_{NN}\left(r,t\right)\right]\frac{r_ir_j}{r^2},$$

where D_{LL} is the longitudinal structure function, and D_{NN} is the transverse structure function. It can be shown:

$$\begin{aligned}\frac{\partial}{\partial r_i}D_{ij} &= \frac{r_j}{r^2}\left[r\frac{\partial D_{LL}}{\partial r}+2\left(D_{LL}-D_{NN}\right)\right]=0,\\ D_{NN} &= D_{LL}+\frac{1}{2}r\frac{\partial}{\partial r}D_{LL}.\end{aligned}$$

Thus, in locally isotropic turbulence $D_{ij}(\boldsymbol{r},t)$ is determined solely by $D_{LL}(r,t)$.

Structure functions (Kolmogorv, 1941b)–contd.

First similarity: For $|\boldsymbol{r}| \ll \mathcal{L}$, D_{ij} is uniquely determined by ϵ and ν,

$$D_{ij}\left(\boldsymbol{r},t\right) = \left(\epsilon r\right)^{2/3} \tilde{D}_{LL}\left(\frac{r}{\eta}\right),$$

where $\tilde{D}_{LL}$ is a dimensionless function (note that $\left(\epsilon r\right)^{2/3}$ has the dimension of vel^2).

Second similarity: For $\eta \ll r \ll \mathcal{L}$,

$$\begin{aligned} D_{LL}\left(r,t\right) &= c_2\left(\epsilon r\right)^{2/3}, \\ D_{NN} &= \frac{4}{3}D_{LL} = \frac{4}{3}c_2\left(\epsilon r\right)^{2/3}, \\ D_{ij}\left(\boldsymbol{r},t\right) &= c_2\left(\epsilon r\right)^{2/3}\left(\frac{4}{3}\delta_{ij} - \frac{1}{3}\frac{r_i r_j}{r^2}\right). \end{aligned}$$

Thus, D_{ij} can be determined by ϵ and c_2.

Structure functions (Kolmogorv, 1941b)–contd.

Saddoughi & Veeravalli (1994): NASA Ames 80 × 120 wind tunnel, $Re_\delta \sim 3.6 \times 10^6$, $\eta = 0.09$ mm, $\mathcal{L}/\eta \approx \delta/\eta \sim 12,000$.

$$
\begin{aligned}
\frac{D_{11}}{(\epsilon)^{2/3}} &= c_2 \sim 2.0 \\
\frac{D_{22}}{(\epsilon)^{2/3}} &= \frac{4}{3} c_2 ?
\end{aligned}
$$

The measured value was within ±15 % of $\frac{4}{3}c_2$.

Karman-Howarth equation

The Kolmogorov hypotheses have no direct connection to the Navier-Stokes equations. It is natural to try to extract useful information about the energy cascade from the Navier-Stokes equations. Karman-Howarth equation is an attempt through two-point correlations.
Consider

$$R_{ij}\left(\boldsymbol{r},t\right)=\overline{u_i\left(\boldsymbol{x}+\boldsymbol{r},t\right)u_j\left(\boldsymbol{x},t\right)}.$$

Note that due to homogeneity, R_{ij} is independent of $\boldsymbol{x}$!. Also note that

$$R_{ij}\left(0,t\right)=\overline{u_iu_j}=\overline{u^2}\delta_{ij}.$$

For homogeneous isotropic turbulence,

$$R_{ij}\left(\boldsymbol{r},t\right)=\overline{u^2}\left[g(r,t)\delta_{ij}+\{f(r,t)-g(r,t)\}\frac{r_ir_j}{r^2}\right],$$

where

$$\begin{aligned} f(r,t) &= \frac{\overline{u_1(\boldsymbol{x}+\boldsymbol{e_1}r,t)u_1(\boldsymbol{x},t)}}{\overline{u_1^2}} \\ g(r,t) &= \frac{\overline{u_2(\boldsymbol{x}+\boldsymbol{e_1}r,t)u_2(\boldsymbol{x},t)}}{\overline{u_2^2}}, \end{aligned}$$

where $f(r,t)$ is the longitudinal auto-correlation and $g(r,t)$ is the transverse (or lateral) auto-correlation.

Karman-Howarth equation–contd.

Note:

$$\begin{aligned} R_{11} &= \overline{u^2} f, \\ R_{22} &= \overline{u^2} g = R_{33}, \\ R_{ij} &= 0 \quad \text{for} \quad i \neq j. \end{aligned}$$

From the continuity equation, $\frac{\partial}{\partial r_j} R_{ij} = 0$, which yields,

$$g(r,t) = f(r,t) + \frac{1}{2} r \frac{\partial f(r,t)}{\partial r}.$$

Thus, $R_{ij}(\boldsymbol{r}, t)$ is a function of $f(r,t)$ only.

Karman-Howarth equation–digression.

Integral scales:

$$\begin{aligned} L_{11} &= L_f = \int_0^\infty f(r)\mathrm{d}r, \\ L_{22} &= L_g = \int_0^\infty g(r)\mathrm{d}r = \frac{1}{2}L_{11}. \end{aligned}$$

Taylor micro scales:

$$\begin{aligned} f(r) &= 1 + \frac{1}{2}f''(0)r^2 + \cdots \\ &= 1 - \left(\frac{r}{\lambda_f}\right)^2. \end{aligned}$$

Thus,

$$\lambda_f = \left[-\frac{1}{2}f''(0)\right]^{-1/2},$$

and similarly

$$\lambda_g = \left[-\frac{1}{2}g''(0)\right]^{-1/2} = \frac{\lambda_f}{\sqrt{2}}.$$

Karman-Howarth equation–digression.

Taylor micro scale and velocity-gradient estimation:

$$
\begin{aligned}
\lim_{r\to 0}\frac{\partial^2}{\partial r^2}\overline{u^2}f(r,t) &= \lim_{r\to 0}\frac{\partial^2}{\partial r^2}\overline{u_1(\boldsymbol{x}+\boldsymbol{e_1}r,t)u_1(\boldsymbol{x},t)} \\
&= \lim_{r\to 0}\overline{\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)u_1} \\
&= \overline{\frac{\partial}{\partial_1}\left(u_1\frac{\partial u_1}{\partial x_1}\right)-\left(\frac{\partial u_1}{\partial x_1}\right)^2} \\
&= -\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}.
\end{aligned}
$$

Thus,

$$
\overline{u^2}f''(0,t) = -\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2},
$$

or

$$
\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \frac{2\overline{u^2}}{\lambda_f^2} = \frac{\overline{u^2}}{\lambda_g^2}.
$$

Karman-Howarth equation–digression.

Note that the estimate of the velocity gradient should be done using the Taylor micro scale.
Recall for isotropic flows,

$$
\begin{aligned}
\epsilon &= 2\nu\overline{s_{ij}s_{ij}} \\
&= \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \\
&= 15\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \\
&= 15\nu\frac{\overline{u_1^2}}{\lambda_g^2}.
\end{aligned}
$$

Taylor's blunder: λ_g is, roughly, a measure of the diameter of the smallest eddies responsible for dissipation.

The above statement is wrong since $\sqrt{\overline{u_1^2}}$ is NOT the characteristic velocity scale of the smallest eddies.

Karman-Howarth equation–digression.

Some comments on Taylor micro scale:

Let $L = k^{3/2}/\epsilon$ be the length scale of energy containing eddies, and

$$Re_L = \frac{k^{1/2}L}{\nu} = \frac{k^2}{\epsilon\nu}.$$

Then

$$\begin{aligned}
\frac{\lambda_g}{L} &= \frac{\left(15\frac{\nu}{\epsilon}\frac{2k}{3}\right)^{1/2}}{L} = \sqrt{10}Re_L^{-1/2} \\
\frac{\eta}{L} &= Re_L^{-3/4} \\
\frac{\eta}{\lambda_g} &= \frac{1}{\sqrt{10}}Re_L^{-1/4}.
\end{aligned}$$

Obviously $\eta < \lambda_g < L$. Although λ_g is mathematically well defined, it does not have a clear physical interpretation.

Karman-Howarth Equation

For homogeneous isotropic flows,

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ij}(\boldsymbol{r}, t) &= \frac{\partial}{\partial t} \overline{u_i(\boldsymbol{x}+\boldsymbol{r}, t)\, u_j(\boldsymbol{x}, t)} \\
&= \overline{u_j(\boldsymbol{x}, t) \frac{\partial}{\partial t} u_i(\boldsymbol{x}+\boldsymbol{r}, t)} + \overline{u_i(\boldsymbol{x}+\boldsymbol{r}, t) \frac{\partial}{\partial t} u_j(\boldsymbol{x}, t)}.
\end{aligned}
$$

Use the Navier-Stokes equations to express $\frac{\partial}{\partial t} u_i(\boldsymbol{x}+\boldsymbol{r}, t)$ and $\frac{\partial}{\partial t} u_j(\boldsymbol{x}, t)$ to get

K-H Equation:

$$
\frac{\partial}{\partial t}\left(\overline{u^2} f\right) - \frac{\overline{u^3}}{r^4} \frac{\partial}{\partial r}\left(r^4 \overline{k}\right) = 2\nu \frac{\overline{u^2}}{r^4} \frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right),
$$

where

$$
\overline{k}(r, t) = \frac{\overline{u_1^2(\boldsymbol{x}, t)\, u_1(\boldsymbol{x}+\boldsymbol{e}_1 r, t)}}{\left(\sqrt{\overline{u^2}}\right)^3}
$$

is the longitudinal triple correlation.

Karman-Howarth Equation–contd.

Note:

$$\begin{aligned}
\lim_{r\to 0} f(r) &= 1, \\
\lim_{r\to 0} \overline{k}(r) &= 0, \\
\lim_{r\to 0} \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right) &= 5f''(0).
\end{aligned}$$

Substituting into the K-H equation,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overline{u^2} &= 10\nu\overline{u^2}f''(0) \\
&= -10\nu\frac{\overline{u^2}}{\lambda_g^2} \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{2}\overline{u^2} &= -15\nu\frac{\overline{u^2}}{\lambda_g^2} \\
\frac{\mathrm{d}k}{\mathrm{d}t} &= -\epsilon,
\end{aligned}$$

i.e., we recover the kinetic energy equation for homogeneous isotropic flows.

Karman-Howarth Equation–contd.

Note:

- The transfer of energy to small scales is accomplished by $\overline{k}$.
- If $\boldsymbol{u}(\boldsymbol{x},t)$ were a Gaussian, field, $\overline{k}=0$ since all 3rd order moments of a Gaussian field is zero. Thus, the energy cascade depends on non-Gaussian aspects of the velocity field.
- Recall

$$
\begin{aligned}
R(r,t) &= \frac{1}{2}\overline{u_k\left(\boldsymbol{x}+\boldsymbol{e}_1 r,t\right)u_k\left(\boldsymbol{x},t\right)} = \frac{1}{2}\overline{u^2}R_{kk}\left(r,t\right) \\
&= \frac{1}{2}\overline{u^2}\left(3g+\frac{f-g}{r^2}r_k r_k\right) \\
&= \frac{1}{2}\overline{u^2}\left(3f+r\frac{\partial f}{\partial r}\right), \\
E(\kappa,t) &= \frac{2}{\pi}\int_0^\infty R(r,t)\ \kappa r\ \sin\kappa r\ \mathrm{d}r.
\end{aligned}
$$

- For a given model for $\overline{k}(r,t)$, solve for $f(r,t)$ from the K-H equation, and then compute the evolution of $R(r,t)$ and $E(\kappa,t)$.
- However, this still does not give a clear picture of the energy cascade. We will directly examine $\frac{\partial}{\partial t}E(\kappa,t)$.

Evolution of the Energy Spectrum

For homogeneous turbulence (see Hinze, 1975),

$$\frac{\partial}{\partial t}E(\kappa,t) = P(\kappa,t) - \frac{\partial}{\partial \kappa}T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t).$$

- $P(k,t)$: production spectrum
  - $P_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} P(\kappa,t)\mathrm{d}\kappa$, contribution to the production from wave numbers (κ_a,κ_b)
  - $P = P_{(0,\infty)} \approx P_{(0,\kappa_{EI})}$, and $P_{(\kappa_{EI},\infty)}/P \ll 1$.
- $T(\kappa,t)$: spectral energy transfer rate, which is the net rate at which energy is transferred from wave numbers lower than κ to larger than κ
  - $\int_{\kappa_a}^{\kappa_b} -\frac{\partial}{\partial\kappa}T(\kappa,t)\mathrm{d}\kappa = T(\kappa_a) - T(\kappa_b)$
  - $\int_0^{\infty} -\frac{\partial}{\partial\kappa}T(\kappa,t)\mathrm{d}\kappa = 0$ since $T(0)=0$ and $T(\infty)=0$.
- $D(\kappa,t) = 2\nu k^2 E(\kappa,t)$: dissipation spectrum, where $\epsilon = \int_0^{\infty} D(\kappa,t)\mathrm{d}\kappa$.

Cascade time scale:

The speed at which the energy travels through the cascade,

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \frac{T_{EI}}{E\left(\kappa\right)} \\ &= \frac{\epsilon}{c\epsilon^{2/3}\kappa^{-5/3}} \\ &= \frac{\kappa^{5/3}\epsilon^{1/3}}{c}.\end{aligned}$$

Thus, $c\epsilon^{-1/3}\kappa^{-5/3}\mathrm{d}\kappa = \mathrm{d}t$. The time it takes from κ_a to κ_b,

$$\begin{aligned}t_{(\kappa_a,\kappa_b)} &= \frac{3}{2}c\epsilon^{-1/3}\left(\kappa_a^{-2/3} - \kappa_b^{-2/3}\right) \\ &= \frac{3}{2}c\tau\left[(\kappa_a L)^{-2/3} - (\kappa_b L)^{-2/3}\right],\end{aligned}$$

where $\tau = L/k^{1/2} = k/\epsilon$ is the eddy turn over time for energy containing eddies .

From the time an eddy entering into the inertial range to the final dissipation,

$$t_{(\kappa_{EI},\infty)} \quad = \quad \tau \frac{3}{2} c \left[(\kappa_{EI} L)^{-2/3} - (\kappa_\infty L)^{-2/3} \right].$$

Note that the second term on the rhs is zero, and

$$\kappa_{EI} \quad = \quad \frac{2\pi}{\ell_{EI}} = \frac{2\pi}{\frac{1}{6} L_{11}} = \frac{2\pi}{\frac{1}{6} 0.4 L}.$$

Thus,

$$\begin{aligned} t_{(\kappa_{EI},\infty)} &= \frac{3}{2} 1.5 \, (30\pi)^{-2/3} \, \tau \\ &\sim \frac{1}{10} \tau. \end{aligned}$$

In other words, the life time of energy once it enters the inertial subrange is just a tenth of its total life time, $\tau = k/\epsilon$.