Fourier Modes

For isotropic turbulence, K-H equation describes the dynamics of the two-point correlation. However, it does not provide a clear picture of the energy cascade process. Further insight can be gained from the N-S equation in the wave space.

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k},t) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\hat{\boldsymbol{u}}$ is the Fourier mode of $\boldsymbol{u}$, and $\boldsymbol{k}$ is a discrete wave number vector.

- Fourier mode is particularly useful for homogeneous turbulence, which can be regard periodic.
- Artificially imposed periodicity in $\mathcal{L}$, where $\mathcal{L} \gg L_{11}$.
  - $\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{N}\mathcal{L}, t) = \boldsymbol{u}(\boldsymbol{x}, t)$
- If $\boldsymbol{u}(\boldsymbol{x},t)$ is real, then $\hat{\boldsymbol{u}}(-\boldsymbol{k},t) = \hat{\boldsymbol{u}}^*(\boldsymbol{k},t)$, where the superscript * denotes a complex conjugate quantity.
- Let $\mathcal{F}\left[g(\boldsymbol{x},t)\right] = \hat{g}(\boldsymbol{k})$
- $\mathcal{F}\left[\frac{\partial}{\partial x_j} g(\boldsymbol{x})\right] = \mathrm{i}k_j \hat{g}(\boldsymbol{k})$
- $\mathcal{F}\left[\nabla^2 g(\boldsymbol{x})\right] = -k^2 \hat{g}(\boldsymbol{k})$, where $k^2 = k_1^2 + k_2^2 + k_3^2$.

Fourier Modes–contd.

- $\mathcal{F}\left[\frac{\partial u_j}{\partial x_j}\right] = \mathrm{i}k_j\hat{h}_j = i\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$, and the continuity equation in the wave space,

$$\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0,$$

  i.e., $\hat{\boldsymbol{u}}$ is perpendicular to $\boldsymbol{k}$.
- Fourier transform of the Navier-Stokes equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_i + \hat{G}_i = -\mathrm{i}k_i\hat{p} - \nu k^2\hat{u}_i,$$

  where $\hat{G}_i(\boldsymbol{k},t)$ is the Fourier transform of the nonlinear term $\frac{\partial}{\partial x_j}u_iu_j$.
- Any vector $\boldsymbol{u}$, it can be decomposed into

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{u}^{\parallel} + \boldsymbol{u}^{\perp}\\ \boldsymbol{u}^{\parallel} &= \boldsymbol{e}(\boldsymbol{e}\cdot\boldsymbol{u}), \quad \boldsymbol{u}^{\perp} = \boldsymbol{u} - \boldsymbol{u}^{\parallel}.\end{aligned}$$

- In Cartesian index notation,

$$\begin{aligned}u_i^{\parallel} &= e_ie_ju_j,\\ u_i^{\perp} &= u_i - e_ie_ju_j\\ &= (\delta_{ij} - e_ie_j)\,u_j\\ &= P_{ij}u_j,\end{aligned}$$

  where P_{ij} denotes the projection tensor (i.e., project a vector into a perpendicular component.

Fourier Modes–contd.

- Poisson equation for pressure

$$
\begin{aligned}
k^2 \hat{p} &= i k_j \hat{G}_j \\
\hat{p} &= \frac{i k_j}{k^2} \hat{G}_j \\
-i k_i \hat{p} &= \frac{k_i k_j}{k^2} \hat{G}_j \\
&= \hat{G}_i^{\|}
\end{aligned}
$$

- Back to the N-S equation,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_i + \hat{G}_i &= -i k_i \hat{p} - \nu k^2 \hat{u}_i \\
&= \hat{G}_i^{\|} - \nu k^2 \hat{u}_i, \\
\frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_i + \hat{G}_i^{\perp} &= -\nu k^2 \hat{u}_i \\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k^2 \right) \hat{u}_i(\boldsymbol{k}, t) &= -\hat{G}_i^{\perp} \\
&= -P_{ij} \hat{G}_j = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \hat{G}_j
\end{aligned}
$$

Energy Spectrum Function, One-Dimensional Spectra

- Once we have $\hat{u}_i(\boldsymbol{k}, t)$, we can compute various statistics:

$$
\begin{aligned}
\hat{R}_{ij}(\boldsymbol{k}) &= \overline{\hat{u}_i(-\boldsymbol{k})\hat{u}_j(\boldsymbol{k})} = \overline{\hat{u}_i^*(\boldsymbol{k})\hat{u}_j(\boldsymbol{k})}, \\
R_{ij}(\boldsymbol{r}, t) &= \sum_{\boldsymbol{k}} \hat{R}_{ij}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \\
\overline{u_i u_j} &= R_{ij}(0, t) = \sum_{\boldsymbol{k}} \hat{R}_{ij}(\boldsymbol{k}, t), \\
\hat{E}(\boldsymbol{k}, t) &= \frac{1}{2}\overline{\hat{u}_i^*(\mathrm{k}, t)\hat{u}_i(\boldsymbol{k}, t)} = \frac{1}{2}\hat{R}_{ii}(\boldsymbol{k}, t), \\
\mathrm{TKE} &= \frac{1}{2}\overline{u_i u_i} = \sum_{\boldsymbol{k}} \frac{1}{2}\hat{R}_{ii}(\boldsymbol{k}, t) = \sum_{\boldsymbol{k}} \frac{1}{2}\hat{E}(\boldsymbol{k}, t), \\
\epsilon(t) &= \sum_{\boldsymbol{k}} 2\nu k^2 \hat{E}(\boldsymbol{k}, t).
\end{aligned}
$$

Energy Spectrum Function, 1-D Spectra–contd.

Recall

$$\Phi_{ij}(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{ij}(\boldsymbol{r}, t)\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\ \mathrm{d}\boldsymbol{r}$$

$$R_{ij}(\boldsymbol{r}, t) = \iiint_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{k}, t)\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\ \mathrm{d}\boldsymbol{k},$$

$$R_{ij}(0) = \overline{u_i u_j} = \iiint_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{k}, t)\ \mathrm{d}\boldsymbol{k}$$

$$\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_\ell}} = \iiint_{-\infty}^{\infty} k_k k_\ell \Phi_{ij}(\boldsymbol{k}, t)\,\mathrm{d}\boldsymbol{k},$$

$$\epsilon = \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} = \iiint_{-\infty}^{\infty} \nu k^2 \Phi_{ij}(\boldsymbol{k}, t)\,\mathrm{d}\boldsymbol{k}.$$

Energy Spectrum Function, 1-D Spectra–contd.

Energy Spectrum Function:

$$
\begin{aligned}
E(k) &= \iiint_{-\infty}^{\infty} \frac{1}{2}\Phi_{ij}\left(\boldsymbol{k},t\right)\delta\left(|\boldsymbol{k}| - \boldsymbol{k}\right)\mathrm{d}\boldsymbol{k}, \\
\frac{1}{2}\overline{u_i u_i} &= \int_0^{\infty} E(k)\mathrm{d}k, \\
\epsilon &= \int_0^{\infty} 2\nu k^2 E(k)\mathrm{d}k.
\end{aligned}
$$

For isotropic turbulence it can be shown,

$$
\Phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) = \frac{E(k)}{4\pi k^2}P_{ij}(\boldsymbol{k}).
$$

Energy Spectrum Function, 1-D Spectra–contd.

- One-Dimensional Spectra, $E_{ij}(k_1)$ is twice the one-dimensional F.T. of $R_{ij}(\boldsymbol{e}_1 r_1)$,

$$E_{ij}(k_1) = 2\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} R_{ij}(\boldsymbol{e}_1 r)1)\mathrm{e}^{-\mathrm{i}k_1 r_1}\mathrm{d}r_1.$$

- For $i = j = 2$, for example,

$$\begin{aligned} E_{22}(k_1) &= \frac{2}{\pi}\int_0^{\infty} R_{22}(\boldsymbol{e}_1 r_1)\cos(k_1 r_1)\mathrm{d}r_1, \\ R_{22}(r_1) &= \frac{2}{\pi}\int_0^{\infty} E_{22}(k_1)\cos(k_1 r_1)\mathrm{d}k_1. \end{aligned}$$

- Note that

$$E_{22}(k_1) = 2\iint_{-\infty}^{\infty} \Phi_{22}(\boldsymbol{k})\mathrm{d}k_2\mathrm{d}k_3,$$

i.e., sum over all k_2 and k_3.

Energy Spectrum Function, 1-D Spectra–contd.

- Similarly,

$$
\begin{aligned}
E_{11}(k_1) &= \frac{2}{\pi}\int_0^\infty R_{11}(\boldsymbol{e}_1 r_1)\cos(k_1 r_1)\mathrm{d}r_1,\\
R_{11}(r_1) &= \frac{2}{\pi}\int_0^\infty E_{11}(k_1)\cos(k_1 r_1)\mathrm{d}k_1.
\end{aligned}
$$

  or using f,

$$
E_{11}(k_1) = \frac{2}{\pi}\overline{u_1^2}\int_0^\infty f(r_1)\cos(k_1 r_1)\mathrm{d}r_1.
$$

- For isotropic turbulence,

$$
\begin{aligned}
E(k) &= \frac{1}{2}k^3\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{k}\frac{\mathrm{d}E_{11}}{\mathrm{d}k}\right),\\
E_{22}(k_1) &= \frac{1}{2}\left(E_{11}(k_1) - k_1\frac{\mathrm{d}E_{11}}{\mathrm{d}k}\right)
\end{aligned}
$$