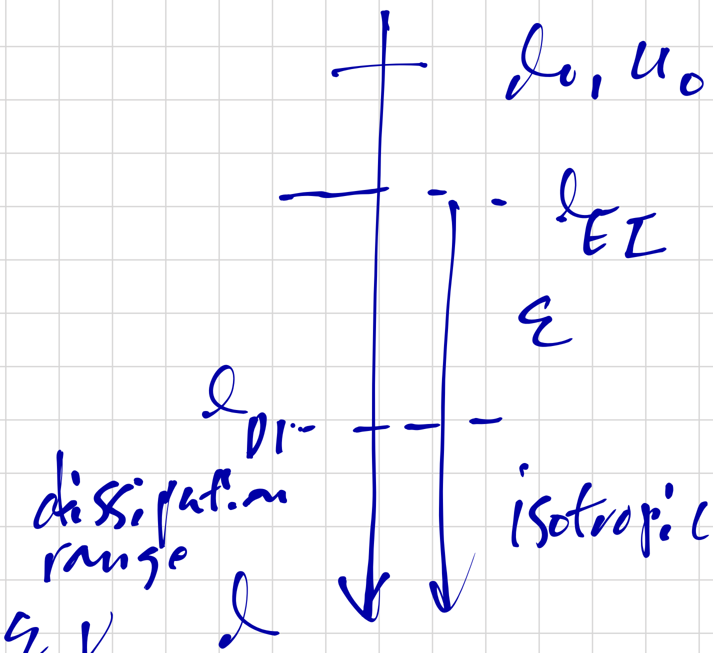
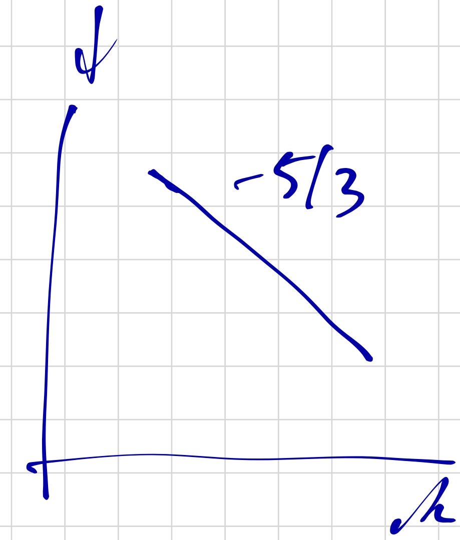
10/16/2018

Recap

Energy cascade



$E_{11}(k_1) \approx \varepsilon^{2/3} k^{-5/3}$



$\eta = (\nu^3/\varepsilon)^{\frac{1}{4}}$

$\eta << \ell << \ell_0$

Kolmogorov (1941b)

$$R_{ij}(\underline{r}) = A\,\delta_{ij} + B\, r_i r_j$$

R_{LL}, R_{NN}

In particular, inertial range

$$R_{ij} = C_2 (\varepsilon/r)^{2/3} \left(\frac{4}{3}\delta_{ij} - \frac{1}{3} \frac{r_i r_j}{r^2} \right) \Leftarrow$$

Ames Exp. $Re_S = 3.6 \times 10^6$

$\delta = 1.09$ mm $\qquad \delta/\eta \sim 12{,}000$

$\eta = 0.09$ mm

$$\frac{R_{11}}{(\varepsilon r)^{2/3}} = C_2 \qquad \frac{R_{22}}{(\varepsilon r)^{2/3}} = \frac{R_{33}}{(\varepsilon r)^{2/3}} = \frac{4}{3} C_2$$

Do K-H here
(see notes)

Evolution of the Energy Spectrum

(Spectral view of energy cascade)

See Hinze (1975), Monin & Yaglom (1975)

For homogeneous turbulence

$$\frac{\partial}{\partial t} E(\kappa, t) = P_\kappa (\kappa, t) - \frac{\partial}{\partial \kappa} T_\kappa (\kappa, t) - \underbrace{2\nu \kappa^2 E(\kappa, t)}_{D(\kappa, t)}$$

Production $\quad$ Transfer $\quad$ Dissipation

$$P_\kappa (\kappa, t) = \int_{\kappa_a}^{\kappa_b} P_\kappa (\kappa, t)\, d\kappa$$

contribution to the production from (κ_a, κ_b)

$$P = \int_0^\infty P_\kappa (\kappa, t)\, d\kappa$$

total production

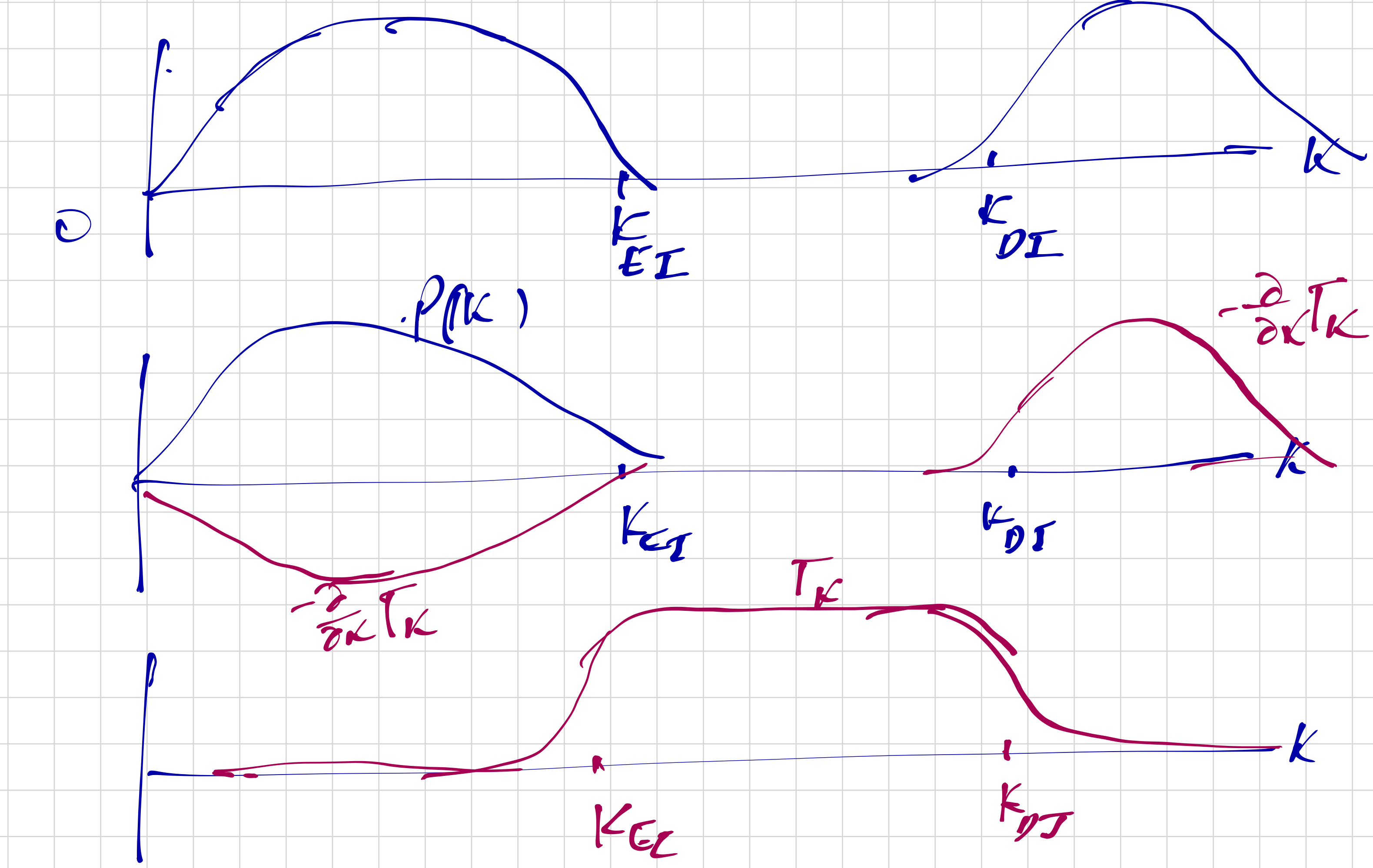
T_κ : spectral energy transfer rate, which is net rate at which energy is transferred from the lower κ to larger κ

$$\int_{\kappa_a}^{\kappa_b} - \frac{\partial}{\partial \kappa} T_\kappa (\kappa, t)\, d\kappa = T_\kappa (\kappa_a) - T_\kappa (\kappa_b)$$

D: Dissipation $\qquad D(\kappa, t) = 2\nu \kappa^2 E(\kappa, t)$

$$\int_0^\infty D(\kappa, t)\, d\kappa = \varepsilon \quad \text{total dissipation}$$

$E(\kappa)$ $\qquad$ $D(\kappa)$



Energy Containing range $(0, K_{EI})$ ✓ TKE

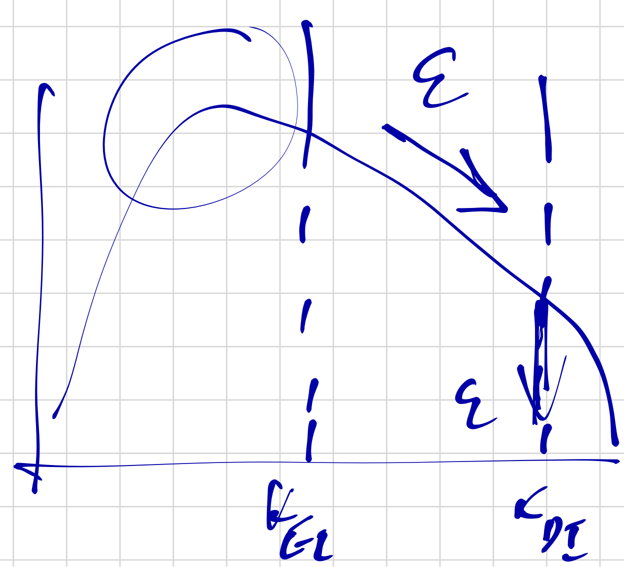
$$\int_0^{K_{EI}} E(k)\,dk \sim \int_0^{\infty} E(k)\,dk = k$$

$$(0, K_{EI}) \qquad \frac{dk}{dt} \sim P - T_{EI}$$

$$(K_{EI}, K_{DI}) \qquad 0 \sim T_{EI} - T_{DI}$$

$$(K_{DI}, \infty) \qquad 0 \sim T_{DI} - \varepsilon$$

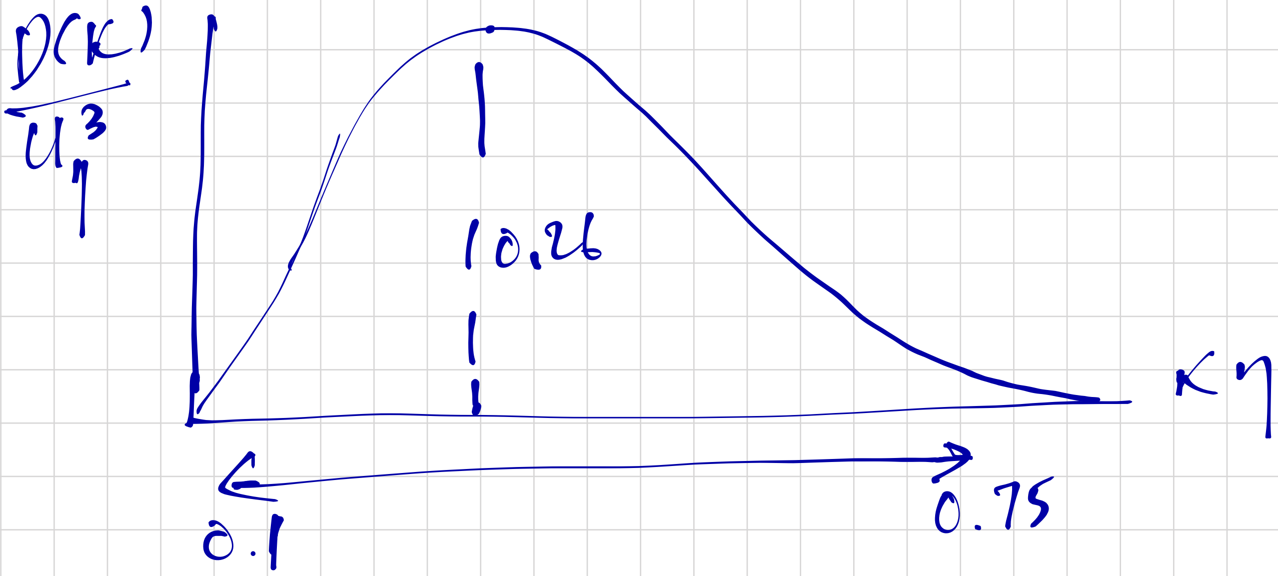
$$\Sigma \quad \frac{dk}{dt} = P - \varepsilon$$



Dissipation Spectrum

$$\varepsilon = \int_0^\infty D(\kappa)\, d\kappa$$

$\uparrow$ $2\nu\kappa^2 E(\kappa)$



$\kappa\eta = 0.1$

$\frac{2\pi}{\ell}\eta = 0.1$

$\frac{\ell}{\eta} \approx 60$

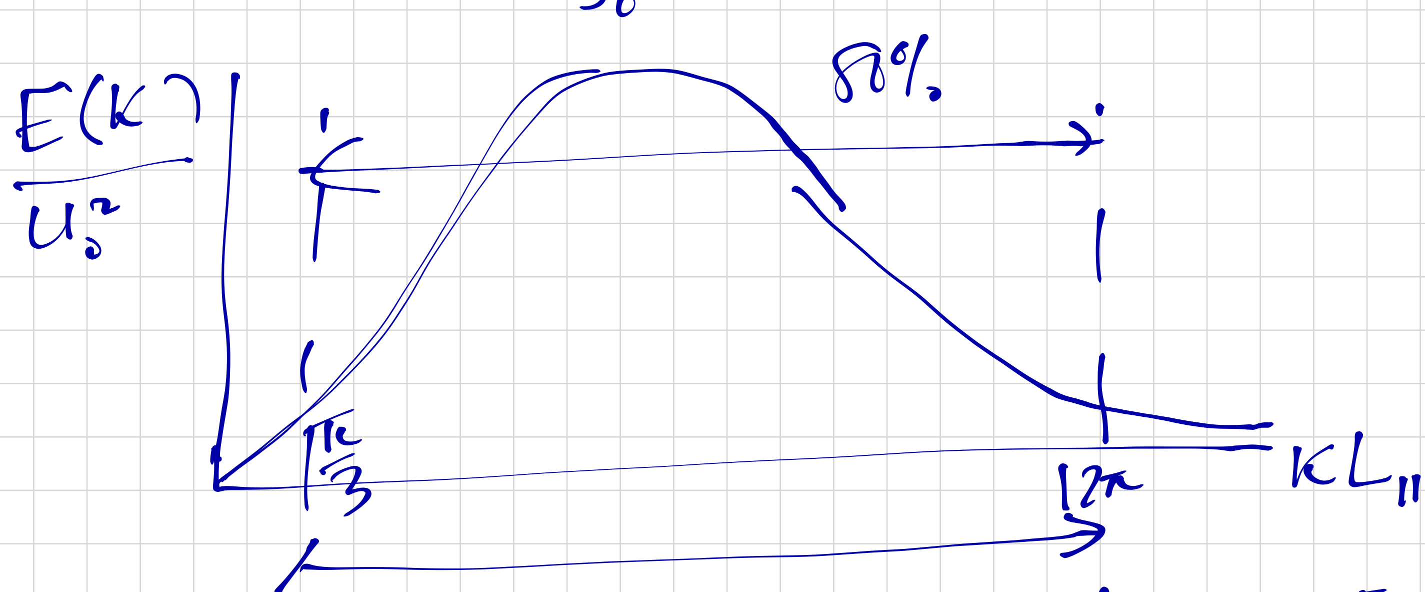
$\ell_{DI} \sim 60\eta$

$\frac{2\pi}{\ell}\eta = 0.75$

$\frac{\ell}{\eta} = 8$

Energy Containing Range

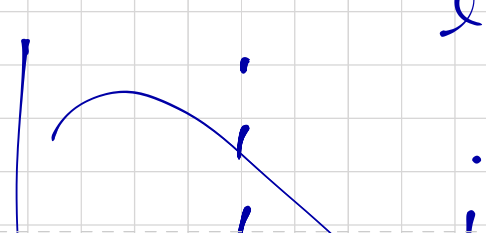
$$k = \int_0^\infty E(\kappa)\, d\kappa$$



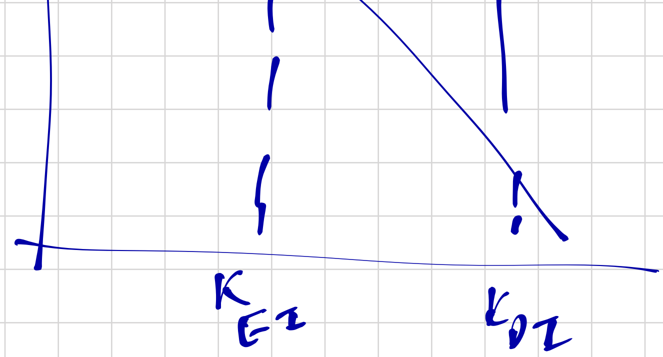
$\kappa L_{11} = 12\pi$

$\frac{2\pi}{\ell} L_{11} = 12\pi \rightarrow \frac{\ell}{L_{11}} = \frac{1}{6}$

$\kappa L_{11} = \frac{\pi}{3}$



$\rightarrow \ell = 6 L_{11}$



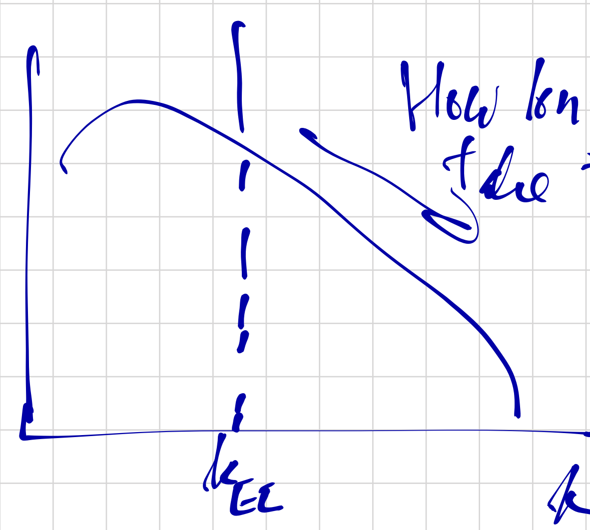
$\ell_{EI} \sim \frac{1}{6}\ell_0 \quad \ell_{DI} \sim 60\eta$

<u>Note</u>

1. Velocity scale for energy containing eddies $\sqrt{k}$
2. Time scale $L_{11}/\sqrt{k} \rightarrow L_{11} = k^{3/2}/\varepsilon$ ✓

   Time scale $\tau = k/\varepsilon \longrightarrow$ Eddy Turn over time
3. Cascade time scale ?

   $\sim \frac{1}{10}\tau$
   (see notes)



How long does it take to get dissipated?

Limitations, Shortcomings and Refinements (Kolmogrov 1962)

1. How large should R_λ ?
2. Energy cascade ; reverse cascade ?
   locally, instantaneously
3. Non-local interaction (i.e. between large and small scale)
4. D_{ij} Higher-order statistics tend to be deviated from Kolmogorov hypotheses

5. Intermittency

6. Refined similarity ε

ε_r : local value of ε, average over $r \ll L$

$\longrightarrow$ stats $\Delta_r u$ depend on ε_r, ν

$\to$ JFM, 1995 JFM 297 Stolovitzky et al.

L, λ, η $\quad L > \lambda > \eta$

$$\varepsilon \sim \left(\frac{\partial u_i}{\partial x_j}\right)^2 \sim \left(\frac{u_\eta}{\eta}\right)^2 \sim \left(\frac{u}{\lambda}\right)^2$$

$$\omega_i \sim \frac{\partial u}{\partial y} \sim \frac{u}{\lambda}$$

What about ~~$\frac{\partial \omega_i}{}$~~ $\frac{\partial \omega_i}{\partial x_j}$?

Vorticity Equation

$$\frac{D}{Dt} \overline{\frac{\omega_i \omega_i}{2}} = \ldots \underbrace{\overline{\omega_i \omega_j s_{ij}}}_{\text{vortex stretching}} - \nu \underbrace{\overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}}}_{\text{viscous dissipation}}$$

For high Re $\quad \overline{\omega_i \omega_j s_{ij}} \sim \nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}}$

$$s_{ij} = O\left(\frac{u}{\lambda}\right), \quad \omega = O\left(\frac{u}{\lambda}\right) \qquad \frac{\partial \omega_i}{\partial x_j} ?$$

$$\left(\frac{u}{\lambda}\right)^3 = \nu \left(\frac{u}{\lambda}\right)^2 \frac{1}{l^2}$$

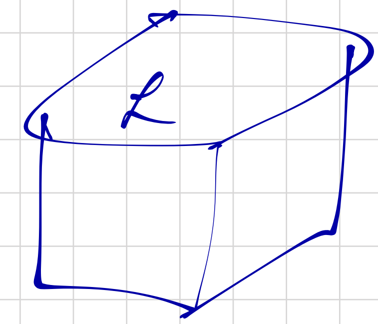
$$l = \left(\frac{\nu \lambda}{u}\right)^{\frac{1}{2}} = \left(\frac{\nu^2 \lambda^2}{u^2}\right)^{\frac{1}{4}} = \left[\frac{\nu^3}{\nu \underbrace{\left(\frac{u}{\lambda}\right)^2}}\right]^{\frac{1}{4}}$$

$$= \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} = \eta$$

Use the Kolmogorov scale to estimate the derivative of ω!

Fourier Modes

$$\underline{u}(\underline{x},t) = \sum_{\underline{k}} \hat{u}(\underline{k},t)\, e^{i\underline{k}\cdot\underline{x}}$$



$L \gg L_{11}$

Let
$$g(\underline{x}) = \sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}\, \hat{g}(\underline{k})$$

$$\mathcal{F}\{g(\underline{x})\} = \hat{g}(\underline{k})$$

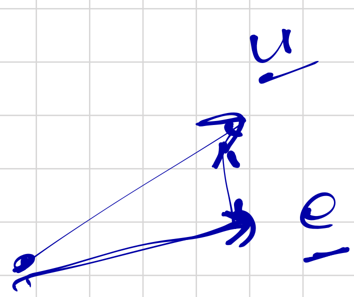
$$\mathcal{F}\left\{\frac{\partial}{\partial x_j} g(\underline{x})\right\} = i k_j \hat{g}(\underline{k})$$

$$\mathcal{F}\{\nabla^2 g\} = -k^2 \hat{g}$$

Conti $\quad \frac{\partial u_i}{\partial x_i} = 0 \qquad \mathcal{F}\left\{\frac{\partial u_i}{\partial x_i}\right\} \rightarrow i k_j \hat{u}_j = 0$

$$\underline{k}\cdot\hat{\underline{u}} = 0$$

$$\hat{\underline{u}} \perp \underline{k}$$



$$\underline{u}^{\parallel} = (\underline{u}\cdot\underline{e})\,\underline{e} \qquad \underline{u}^{\perp} = \underline{u} - \underline{u}^{\parallel}$$

$$u_i^{\parallel} = (u_j e_j)\, e_i \qquad u_i^{\perp} = u_i - (u_j e_j) e_i$$

$$= u_j \delta_{ij} - u_j e_j e_i$$

$$= u_j(\delta_{ij} - e_i e_j)$$

$$\underline{e} = \underline{k}/|\underline{k}|$$

$$e_i e_j = \frac{k_i k_j}{k^2}$$

$$P_{ij}$$

$$P_{ij} = \left(\delta_{ij} - k_i k_j / k^2\right)$$

Projection tensor

$$\mathcal{F}\{N\text{-}S\}$$

$$\mathcal{F}\left\{ \frac{\partial u_i}{\partial t} + \underbrace{\frac{\partial}{\partial x_j} u_i u_j}_{G_i} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right\}$$

$$\frac{\partial}{\partial t} \hat{u}_i + \hat{G}_i = \underbrace{-i k_i \hat{p}} - \nu k^2 \hat{u}_i$$

Poisson equ. for pressure

$$k^2 \hat{p} = i k_j \hat{G}_j$$

$$\hat{p} = i \frac{k_j}{k^2} \hat{G}_j$$

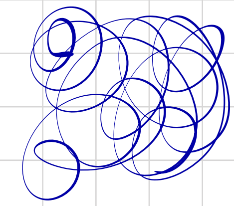
$$-i k_i \hat{p} = \frac{k_i k_j}{k^2} \hat{G}_j = G_i^{\parallel}$$

$$\frac{\partial}{\partial t} \hat{u}_i + \underbrace{\hat{G}_i}_{\hat{G}_i^{\perp} + \hat{G}_i^{\parallel}} = \hat{G}_i^{\parallel} - \nu k^2 \hat{u}_i$$

$$\frac{\partial \hat{u}_i}{\partial t} = - G_i^{\perp} - \nu k^2 u_i$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{u}_i = - G_i^{\perp} = - \underbrace{\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)} \hat{G}_j$$

$$\frac{D}{Dt} \overset{\checkmark}{k} = P + T - \varepsilon$$

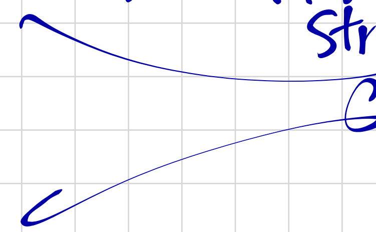


Isotropic turbulence

$$\frac{D}{Dt} k = P + T - \varepsilon$$

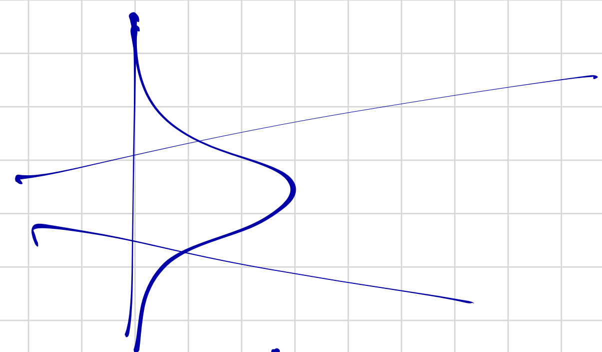
Homog. Shear

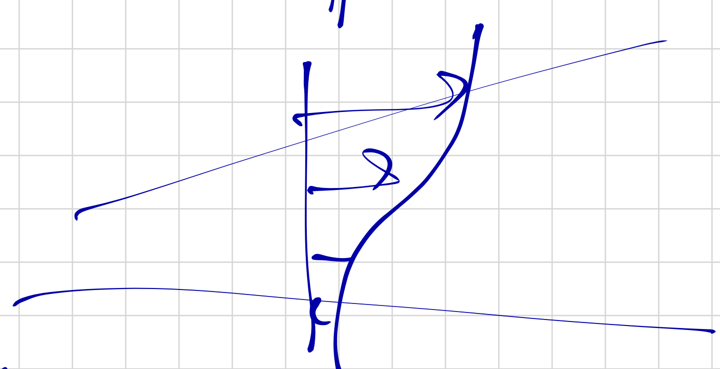
Homog. Strain



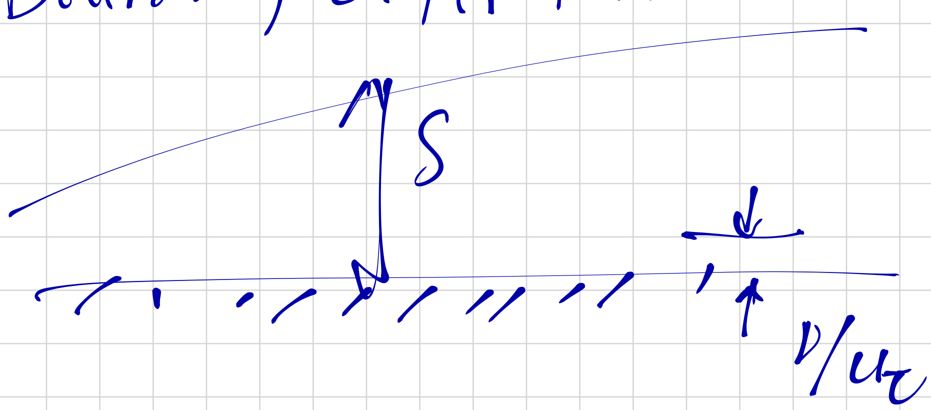
Inhomog. Flows

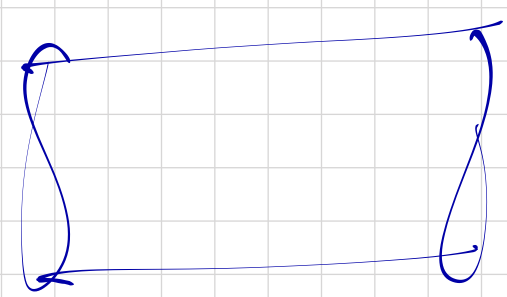
Free-Shear Flows

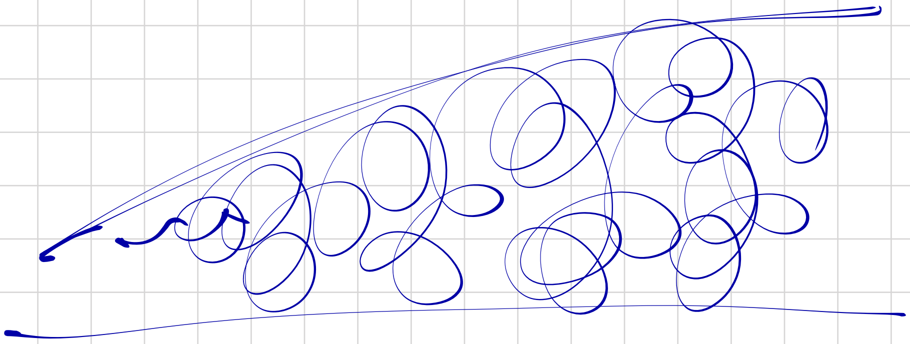




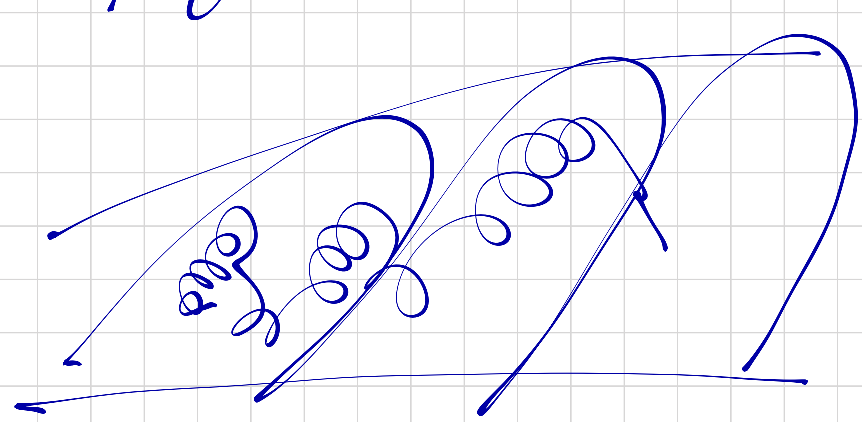
Boundary Layer Flows







Old view



New View