10/11/2018

Recap

RANS: $\tilde{u}_i = U_i + u_i$

$$\frac{D}{Dt} U_i = \dots \frac{\partial}{\partial x_j}\overline{u_i u_j} + \dots$$

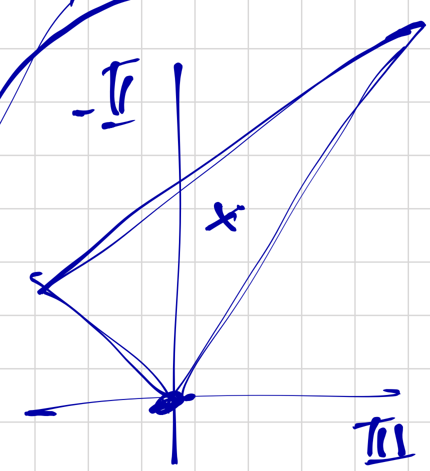
$$\frac{D}{Dt}\overline{u_i u_j} = \dots \frac{\partial}{\partial x_k}\overline{u_i u_j u_k} + \dots$$

Turbulence Closure Problem → Model

$$\overline{u_i u_j} - \frac{2}{3}k\,\delta_{ij} = 2\nu_T S_{ij}$$

$\overline{u_i u_j}$

$\underline{b_{ij}}$ $\quad$ I, II, III



$$\frac{D}{Dt}\underbrace{\frac{\overline{u_i u_i}}{2}}_{k} = \underbrace{-\overline{u_i u_j}S_{ij}}_{\text{Production terms}} + \dots$$

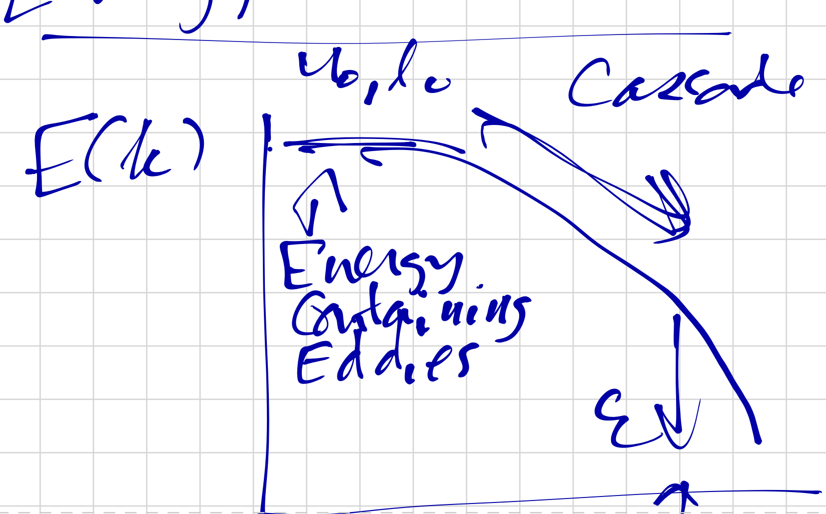
Simple Shear Flows

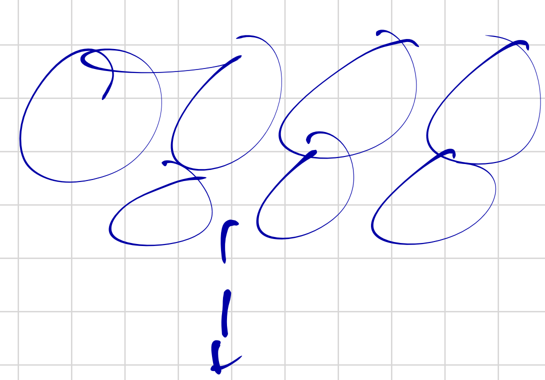
$$\frac{\partial}{\partial t}\overline{u_1^2} = -\overline{u_1 u_2}\frac{\partial U_1}{\partial x_2} + \overline{p\frac{\partial u_1}{\partial x_1}} + \dots$$

$$\overline{u_2^2} = 0 + \overline{p\frac{\partial u_2}{\partial x_2}} + \dots$$

$$\overline{u_3^2} = 0 + \underbrace{\overline{p\frac{\partial u_3}{\partial x_3}}}_{\text{Inter component transfer}}$$

Energy Cascade





0.0 $\quad 10^2 \quad 10 \quad 1$ $\quad \frac{u\ell}{\nu}$

k

Dissipating Eddies

$\varepsilon \sim u_0^3/l_0$

Local Isotropy

Small scale $\quad l \ll l_0 \quad$ isotropic

$l < l_{EI}$ isotropic

$l > l_{EI}$ anisotropic

$\tau_{small} \ll \tau_{energy\ containing}$

$\frac{dk}{dt} = -\varepsilon$

Kolmogorov 1st Similarity

During the cascade process, all information about the geometry of large eddies is lost

→ Statistics of small scale have a universal form, i.e., similar, in any high Re flows, and it is uniquely determined by ν and ε. Small scales receive energy from large eddies, and dissipates the energy.

$l < l_{EI}$ is referred to as universal equilibrium range.

$\tau(l) = \frac{l}{u(l)}$ is small (compared to l_0/u_0)

such that small eddies adapt quickly to maintain dynamic equilibrium with the maintain a dynamic equilibrium with the energy transfer imposed by the large eddies.

$$\eta = \nu^a \varepsilon^b = \left(\frac{L^2}{T}\right)^a \left(\frac{L^2}{T^3}\right)^b$$

$$\left.\begin{aligned} L:&\quad 2a+2b=1 \\ T:&\quad -a-3b=0 \end{aligned}\right] \Rightarrow a=\frac{3}{4},\ b=-\frac{1}{4}$$

$$\boxed{\begin{aligned} \eta &= \left(\nu^3/\varepsilon\right)^{\frac{1}{4}} \\ u_\eta &= (\varepsilon\nu)^{1/4} \\ \tau_\eta &= (\nu/\varepsilon)^{\frac{1}{2}} \end{aligned}}$$

Kolmogorov length scales.

On the average, the smallest length scale in turbulent flows.

Note : 1. $Re_\eta = \frac{u_\eta \eta}{\nu} = 1$

2. $\varepsilon = 2\nu \overline{S_{ij}S_{ij}}$

$$\sim \nu\left(\frac{u_\eta}{\eta}\right)^2 = \nu/\tau_\eta^2$$

$\frac{u_\eta}{\eta} = \frac{1}{\tau_\eta}$ time scale associated with the dissipation

"Similarity" & "universal"

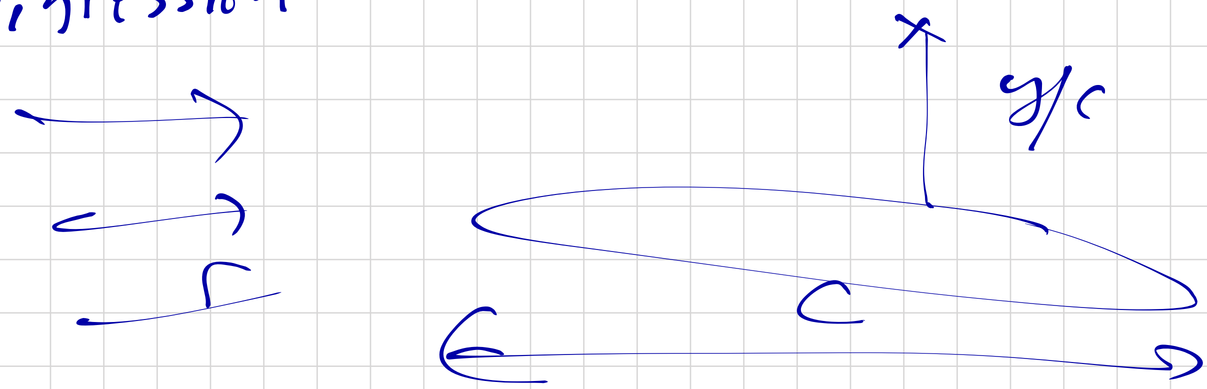
$\underline{y} = (\underline{x} - \underline{x}_0)/\eta$, $\underline{w}(\underline{y}) = [\underline{u}(\underline{x}) - \underline{u}(\underline{x}_0)]/u_\eta$

For $|\underline{y}| < \frac{l_{EI}}{\eta}$, $\underset{\uparrow}{\varepsilon}$ $\underset{\uparrow}{\nu}$

$\varepsilon \stackrel{D}{=} \frac{L^2}{T^3}$

$\nu \stackrel{D}{=} L^2/T$

Digression

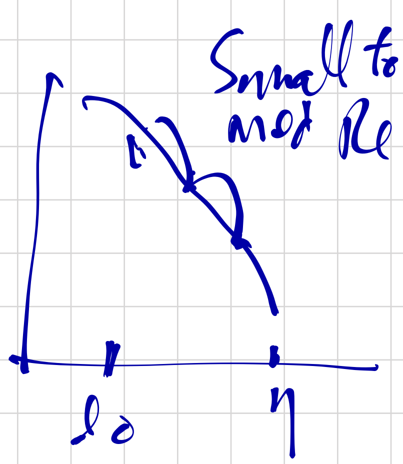


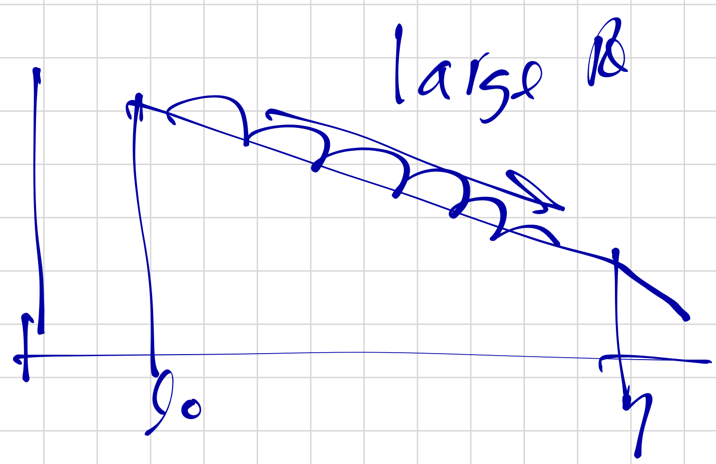
Statistics indep. of ε and ν (∵ cannot construct a non-dim out of ε and ν)

→ Similar!

Also
$$\frac{\eta}{l_0} = \left(\frac{\nu^3}{\varepsilon\, l_0^4}\right)^{\frac{1}{4}} = \left(\frac{\nu^3}{u_0^3/l_0^3}\right) = \left(\frac{\nu}{u_0 l_0}\right)^{-3/4} = Re^{-3/4}$$

$$\frac{l_0}{\eta} = Re^{3/4}$$





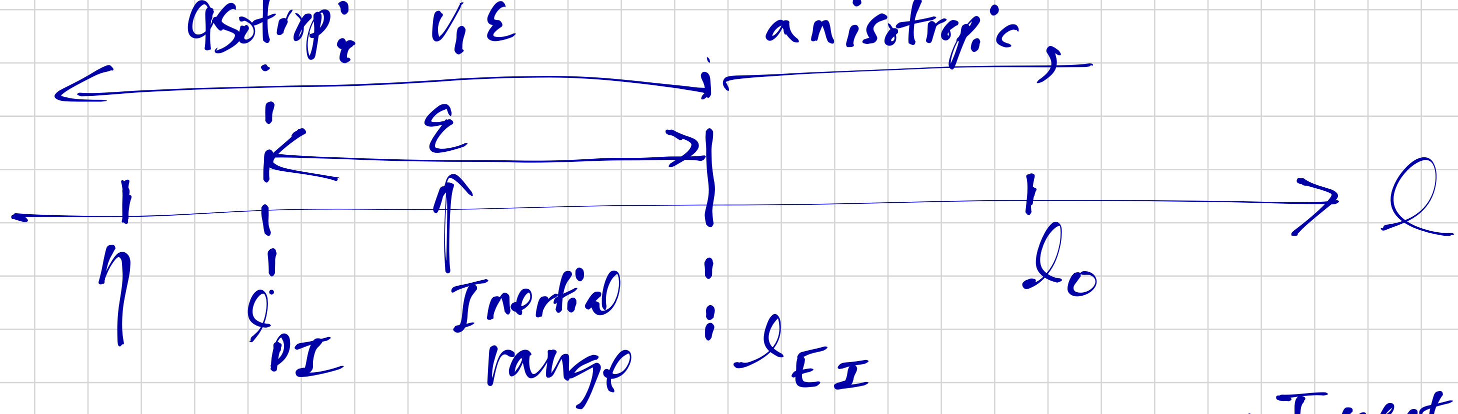
Similarly
$$\frac{u_\eta}{u_0} \sim Re^{-\frac{1}{4}}, \qquad \frac{\tau_\eta}{\tau_0} \sim Re^{-\frac{1}{2}}$$

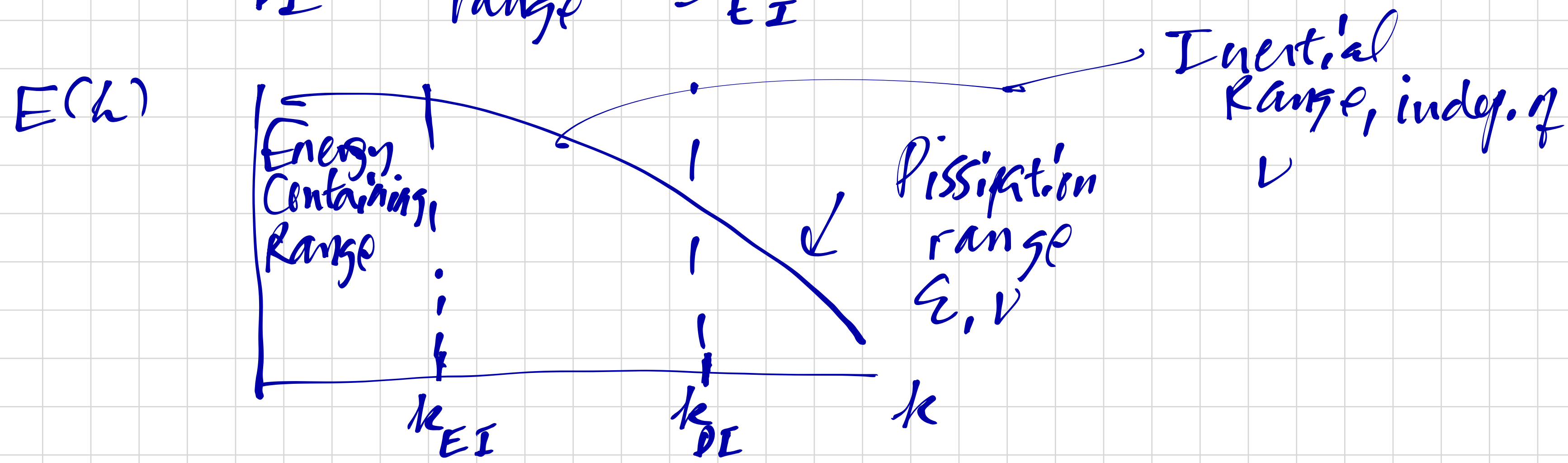
$\boxed{\text{Show images for large Re}}$

Kolmogorov 2nd Similarity

At sufficiently large Re $\quad \eta \ll l \ll l_0$

the statistics of the motion on ℓ
have a universal form indep. of ν





Inertial range

$$u(\ell) = (\varepsilon \ell)^{1/3} = u_\eta \left(\frac{\ell}{\eta}\right)^{1/3} = u_0 \left(\frac{\ell}{\ell_0}\right)^{1/3}$$

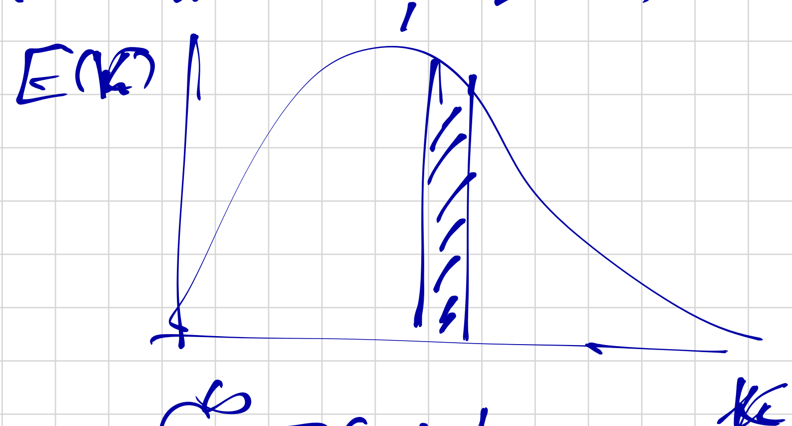
$$\tau(\ell) = (\ell^2/\varepsilon)^{1/3} = \tau_\eta \left(\frac{\ell}{\eta}\right)^{2/3} = \tau_0 \left(\frac{\ell}{\ell_0}\right)^{2/3}$$

Thus $\ell \downarrow \quad u(\ell), \tau(\ell) \downarrow$

The energy transfer rate T_{EI} in the inertial range

$$T_{EI} \sim \frac{u(\ell)^2}{\tau(\ell)} = \frac{(\varepsilon \ell)^{2/3}}{(\ell^2/\varepsilon)^{1/3}} = \varepsilon$$

Energy Spectrum Function, $E(\kappa)$ $\quad \kappa = |\underline{\kappa}|$



$$TKE = k = \frac{1}{2}\overline{u_i u_i} = \int_0^\infty E(\kappa)\, d\kappa$$

$$E(\kappa) \overset{D}{=} Vel^2 L = L^3/T^2$$

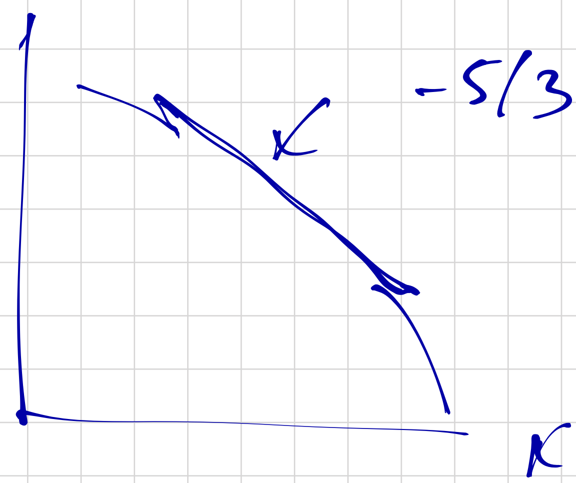
$$E(\kappa) = C\,\varepsilon^{m}\,\kappa^{n}$$

$$\rightarrow m = 2/3 \qquad n = -5/3$$

$$\boxed{E(\kappa) = C\,\varepsilon^{2/3}\,\kappa^{-5/3}}$$ "−5/3 law"

~1.5 Kolmogorov Const.

$E(\kappa)$



Recall 1-D Spectra

$$\Phi_{ij}(\underline{k}) = \ldots$$

$$\begin{cases} \phi_{11} = \dfrac{1}{2\pi}\displaystyle\int R_{11}(r_1, 0, 0)\, e^{-ik_1 r_1}\, dr_1 \\ R_{11} \end{cases}$$

In general, the relationship between ϕ_{11} and $E(\kappa)$ very complicated. For isotropic turbulence,

$$E_{11} \triangleq 2\phi_{11}(k_1)$$

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left(\frac{1}{\kappa}\frac{dE_{11}}{d\kappa}\right)$$

$$E_{22}(k_1) = \frac{1}{2}\left(E_{11} - k_1 \frac{d}{dk_1} E_{11}\right)$$

$$E(\kappa) \sim \kappa^{m} \longrightarrow E_{11}, E_{22} \sim k_1^{m}$$

Inertial range

$$E(k) = C\,\varepsilon^{2/3}\,k^{-5/3}$$
$$E_{11}(k_1) = c_1\,\varepsilon^{2/3}\,k_1^{-5/3}$$
$$E_{22}(k_1) = c_2\,\varepsilon^{2/3}\,k_1^{-5/3}$$

all have "$-5/3$" slope

$C \sim 1.5$

$c_1 \sim 0.49$ $\qquad c_2 = \frac{4}{3} c_1 \sim 0.65$

So $E_{22} = \frac{4}{3} E_{11}$

evidence for isotropy

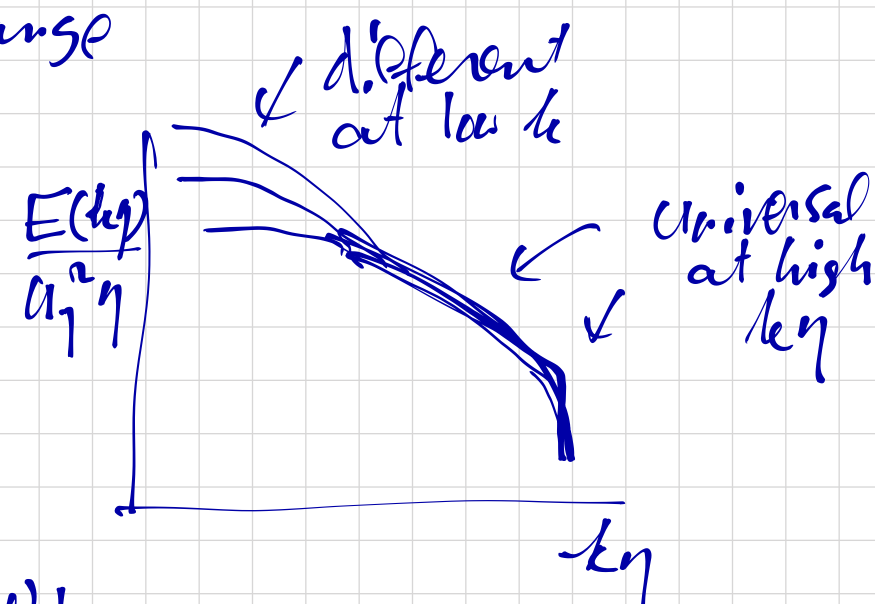
Isotropic Flows: $\left(\frac{\partial u_1}{\partial x_1}\right)^2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}\right)^2$

Only true when all scales are isotropic

At high k, dissipation range

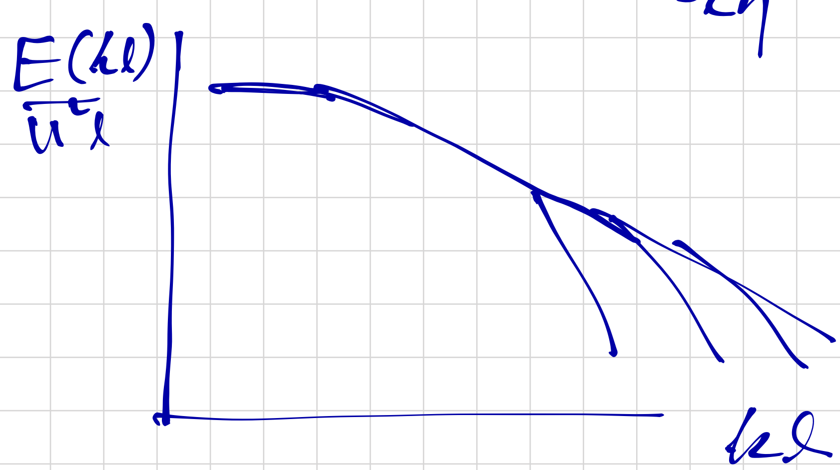
$$E(k) = E(k;\varepsilon,\nu)$$

$$\frac{E(k)}{u_\eta^2\,\eta} = f(k\eta)$$



At low k

$$\frac{E(k)}{u^2\,l} = F(kl)$$



In the overlap region

$$\frac{E}{u^2 l} = B\,(kl)^{-5/3}$$

$$E(kl) = B\,u^2\,l\,k^{-5/3}\,l^{-5/3}$$

$$= B\left(\frac{u^3}{l}\right)^{2/3} k^{-5/3} \quad ☆$$

If you recall

$$E(k) = C\, \varepsilon^{2/3}\, k^{-5/3}$$

From ~~...~~, $\longrightarrow$ $\boxed{\varepsilon = u^3/\ell}$ indep. of ν

Rough estimation ε

TBL $\quad \varepsilon \sim u_{rms}^3/\delta$

Airbus cruising at 10,000 meter high.

$M_\infty \sim 0.85 \qquad M_\infty = U_\infty / C$

$\nu = 3.5 \times 10^{-5}\ m^2/s$

$C = 290\ m/s$

$\rightarrow U_\infty = 250\ m/s \quad (\sim 560\ mph)$

$$Re = \frac{U_\infty C_h}{\nu} = \frac{250\ m/s\ 5\ m}{3.5\times10^{-5}\ m^2/s} = 3.5\times10^{7}$$

$$\delta = 0.37\, \ell\, (Re)^{-0.2} \sim \boxed{5.7\times10^{-2}\ m}$$ Integral scale

$$\varepsilon \sim u^3/\ell = (25\ m/s)^3 / 5.7\times10^{-2}\ m = 2.74 \times 10^5\ m^2/s^3$$

$\frac{u_{rms}}{U_\infty} \sim 0.1 \qquad u_{rms} = 25\ m/s$

$$\eta = (\nu^3/\varepsilon)^{1/4} = \boxed{2\times10^{-5}\ m}$$ Kolmogorov scale

$$\varepsilon = 15\,\nu \left(\frac{\partial u_1}{\partial x_1}\right)^2 \sim 15\,\nu \left(\frac{u}{\eta}\right)^2$$

$$\lambda = \left(\frac{15 \nu u^2}{\varepsilon}\right)^{\frac{1}{2}} = \boxed{10^{-3}\ m}$$ Taylor Micro Scale

$$\eta < \lambda < \delta$$

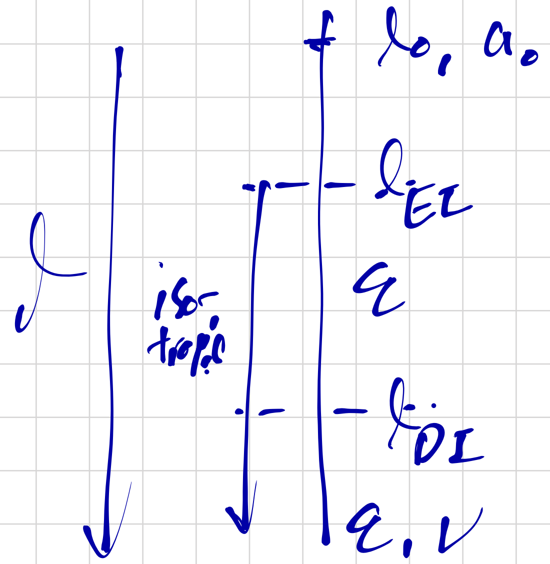
TBL $\frac{\nu}{u_\tau}$

$$u_\tau = \sqrt{\frac{C_f}{2} U_\infty^2}$$ $\Longrightarrow$ $\frac{\nu}{u_\tau} = \boxed{4 \times 10^{-6}\ m}$

Viscous wall scale

$\sim$ Kolmogorov Scale

$$C_f = \frac{0.455}{(\log Re)^{2.58}} = 2.48 \times 10^{-3}$$



$\frac{u_0 l_0}{\nu} \gg 1$

1941

1941b

Structure Function (Kolmogorov 1941b)

$$D_{ij}(\underline{r}, \underline{x}, t) = \overline{[\tilde{u}_i(\underline{x}+\underline{r}, t) - \hat{u}_i(\underline{x}, t)][\hat{u}_j(\underline{x}+\underline{r}, t) - \hat{u}_j(\underline{x}, t)]}$$

Local isotropy $|\underline{r}| \ll \mathcal{L}$

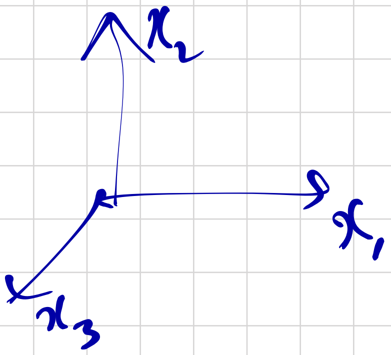
$D_{ij}(\underline{r}) \longrightarrow$ isotropic

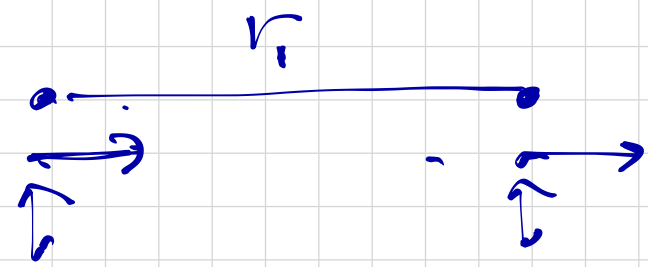
$$D_{ij} = A\,\delta_{ij} + B\,r_i r_j$$

General 2^{nd}-order isotropic tensor

D_{LL} : longitudinal structure function

D_{NN} : transverse 〃





$D_{11} = D_{LL}$

$D_{22} = D_{NN}$

$$D_{ij} = \underbrace{D_{NN}}_{A}\,\delta_{ij} + \underbrace{\left[D_{LL} - D_{NN}\right]\frac{1}{r^2}}_{B}\, r_i r_j$$

Note

$$\frac{\partial}{\partial r_i} D_{ij} = 0$$

$$\frac{\partial}{\partial r_i} D_{ij} = \frac{r_j}{r^2}\left(r \frac{\partial}{\partial r} D_{LL} + 2\left(D_{LL} - D_{NN}\right)\right) = 0$$

$$\longrightarrow \quad D_{NN} = D_{LL} + \frac{1}{2} r \frac{\partial}{\partial r} D_{LL}$$

$\longrightarrow$ D_{ij} is a function of D_{LL} alone for isotropic turbulence

$$D_{ij} = (\varepsilon r)^{2/3} D_{LL}\left(\frac{r}{\eta}\right) \qquad 1^{st} \text{ Similarity}$$

Inertial Range

$$D_{ij} = C_2 (\varepsilon r)^{2/3} \left(\frac{4}{3}\delta_{ij} - \frac{1}{3}\frac{r_i r_j}{r^2}\right)$$

Saddoughi & Veeravalli (JFM, 1994)

$Re_\delta = 3.6 \times 10^6$ $\quad \delta = 1.09$ mm $\quad \frac{\delta}{\eta} \simeq 12{,}000$

$\eta = 0.09$ mm

$$D_{11}/(\varepsilon r)^{2/3} = C_2 \to 2.0$$

$$P_{22}/(\varepsilon r)^{2/3} = \frac{4}{3} C_2$$